Chapter 2 Notes.

Review of Quantum Mechanics & Special Relativity

1.) Units

1 barn $= 10^{-24}\ \mathrm{cm}^2$

Natural Units $\rightarrow$ $\hbar = c = 1$

$$E^2 = p^2c^2 + m^2c^4 \quad \rightarrow \quad E^2 = p^2 + m^2$$

Energy, Momentum, Mass $\rightarrow$ [GeV]

Time, Length $\rightarrow$ $[\mathrm{GeV}^{-1}]$

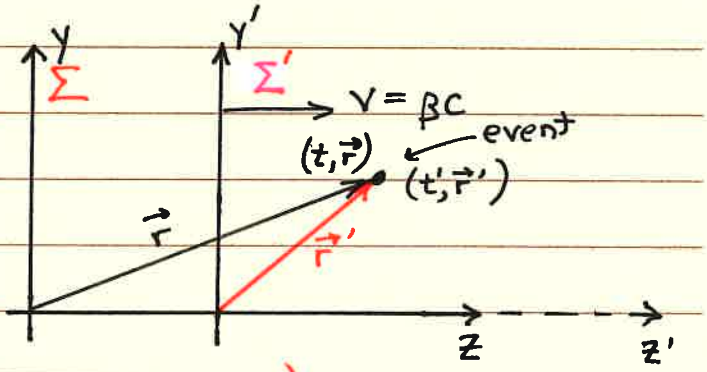
Area (e.g. the barn) $\rightarrow$ $[\mathrm{GeV}^{-2}]$

$\hbar c = 0.197$ GeV·fm

Also $\varepsilon_0 = \mu_0 = 1$ Heaviside - Lorentz units

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = \frac{1}{137} \quad \longrightarrow \quad \alpha = \frac{e^2}{4\pi}$$

2.) Lorentz Transformation



The space-time interval

wave front of
a light pulse

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

is an invariant, because c (the speed of light) = constant in all inertial frames. (Einstein)

X (a 4-vector)

X′ (a 4-vector)

$$\begin{pmatrix} \underline{\underline{c}}t' \\ x' \\ y' \\ z' \end{pmatrix} = \left(\begin{array}{cc|cc} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{array}\right) \begin{pmatrix} \underline{\underline{c}}t \\ x \\ y \\ z \end{pmatrix}$$

$$\underline{\underline{c}}t' = \gamma(ct - \beta z)$$
$$x' = x$$
$$y' = y$$
$$z' = \gamma(z - \beta \underline{\underline{c}}t)$$

$$X' = \Lambda X$$

Show $\Lambda\Lambda^{-1} = I$

The inverse transform $(\beta \rightarrow -\beta)$ $\quad X = \Lambda^{-1} X'$

Chapter 2 Notes

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \Lambda^{\mu}{}_{\nu} = \Lambda$$

$$x'_{\mu} = \Lambda_{\mu}{}^{\nu} x_{\nu} \qquad \Lambda_{\mu}{}^{\nu} = \Lambda^{-1}$$

$$\begin{pmatrix} t' \\ -x' \\ -y' \\ -z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} t \\ -x \\ -y \\ -z \end{pmatrix}$$

$x^{\mu} = (t, x, y, z)$ contravariant

$x_{\mu} = (t, -x, -y, -z)$ covariant

$x^{\mu} x_{\mu} = t^2 - x^2 - y^2 - z^2$ the Lorentz-Invariant Space-Time interval.

Covariant ⟷ Contravariant

$$x_{\mu} = g_{\mu\nu} x^{\nu} \quad \text{where} \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$a^{\mu} b_{\mu} = a_{\mu} b^{\mu} = g_{\mu\nu} a^{\mu} b^{\nu}$ automatically a Lorentz-Invariant

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Four-Momentum – p^{μ}

Recall:

Classical	Relativistic
* $E = mc^2$ (slow)	$E = \gamma mc^2$
$p = mv$ (slow)	$p = mc\beta\gamma$

m = rest mass

The momentum 4-vector $p^{\mu} = (E, p_x, p_y, p_z)$

$p_{\mu} = (E, -p_x, -p_y, -p_z)$

$$p^{\mu} p_{\mu} = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$E^2 - \vec{p}\cdot\vec{p} = E^2 - \vec{p}^{\,2} = m^2$$ the rest-mass squared.

A System of Particles $\overset{\text{TOTAL}}{P^{\mu}} = p_1^{\mu} + p_2^{\mu} + \dots = (E_1 + E_2 + \cdots, \vec{p}_1 + \vec{p}_2 + \cdots)$

Chapter 2 Notes

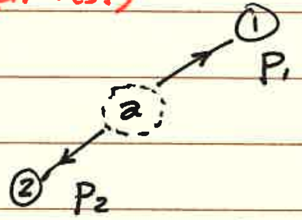
$$\overset{\text{Total}}{P^\mu} = p_1^\mu + p_2^\mu + \cdots = \left(\sum E_i , \sum \vec{p}_i\right)$$
$$= \left(\sum E_i , \sum p_{x_i} , \sum p_{y_i} , \sum p_{z_i}\right)$$

$$s = (\text{CM energy})^2 = \overset{\text{Total}}{P^\mu} \overset{\text{Total}}{P_\mu} = s$$

Two-Particle Decay

$$a \to 1 + 2 \qquad \overset{\text{Total}}{P^\mu} = p_1 + p_2$$

m_a (at rest)



$$\overset{\text{TOTAL}}{P^\mu} = P^\mu = p_1^\mu + p_2^\mu$$
$$P^\mu P_\mu = (p_1 + p_2)^\mu (p_1 + p_2)_\mu$$
$$= m_a^2$$

↑ "true", even if m_a were moving before it decays.

Four-Derivative

worked out in detail in our textbook Thomson

$$\begin{pmatrix} \partial/\partial t' \\ \partial/\partial x' \\ \partial/\partial y' \\ \partial/\partial z' \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & 0 & 0 & +\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +\beta\gamma & 0 & 0 & \gamma \end{pmatrix}}_{\Lambda^{-1}} \begin{pmatrix} \partial/\partial t \\ \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}$$

must be covariant

$$= \partial_\mu = \left(\partial/\partial t , \partial/\partial x , \partial/\partial y , \partial/\partial z\right) \Rightarrow \partial_\mu = \frac{\partial}{\partial x^\mu}$$

This seems "backwards" but it is correct.

Likewise,

$$\partial^\mu = \left(\partial/\partial t , -\partial/\partial x , -\partial/\partial y , -\partial/\partial z\right) \Rightarrow \partial^\mu = \frac{\partial}{\partial x_\mu}$$

d'Alembertian

$$\Box = \partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

$\Box^2$

Chapter 2 Notes

Vector and Four-Vector Notation

1.) Three-vectors $\vec{p}_1 \cdot \vec{p}_2 = |\vec{p}_1||\vec{p}_2| \cos\theta$

not squared

2.) Four-vector $p_1 \cdot p_2 = p_1^{\mu} p_{2\mu} = (p_1^0 p_2^0 - p_1^1 p_2^1 - p_1^2 p_2^2 - p_1^3 p_2^3)$

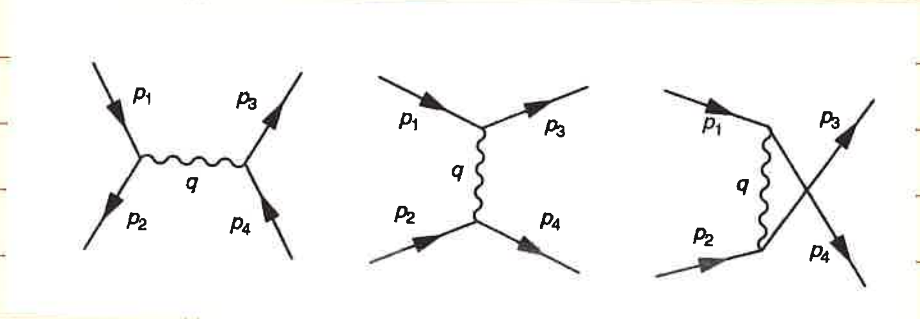
$$\boxed{p_1 \cdot p_2 = E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2}$$

This is true in general for any pair of Four-Vectors.

... is a Lorentz-Invariant

Mandelstam Variables

2 particles → 2 particles



$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2$$

$*$ = measured in CM system

In the CM System

$$s = (p_1 + p_2)^2 = (E_1^* + E_2^*)^2 - (\underbrace{\vec{p}_1^{\,*} + \vec{p}_2^{\,*}}_{=0 \text{ in CM system}})^2 = (E_1^* + E_2^*)^2$$

Chapter 2 Notes

Non-Relativistic Quantum Mechanics

Relativistic treatment of Spin-½ particles → Chapter 4

Free particle is represented by a plane wave

$$\psi(\vec{x},t) \sim e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

However, $\vec{p} = \hbar\vec{k}$ and $E = \hbar\omega$

↑ wave vector ↑ ang. frequency

Free-particle → $\psi(x,t) = N e^{i(\vec{p}\cdot\vec{x}-Et)}$

↑ Normalization constant

Note: The time-dependent variables of classical mechanics ⟹ Time-independent operators acting on time-dependent wavefunctions

Are replaced by

A physical observable quantity A corresponds to the action of a QM operator $\hat{A}$ on a wavefunction

$$\hat{A}\psi = a\psi$$

↑ result of the measurement will be one of the eigenvalues of this equation.

The eigenvalues must be "real" ⟹ $\hat{A}$ must be Hermitian.

In other words: $\int \psi_1^* \hat{A} \psi_2 \, d^3r = \int [\hat{A}\psi_1]^* \psi_2 \, d^3r$

For a plane wave: $\psi(\vec{x},t) = N e^{i(\vec{p}\cdot\vec{x}-Et)}$

$\hat{p} = -i\vec{\nabla}$ and $\hat{E} = i\,\partial/\partial t$

Chapter 2 Notes

Total Energy $E = H = \overset{\text{K.E.}}{T} + \overset{\text{P.E.}}{V} = \frac{P^2}{2m} + V$

non-relativistic

Non-Relativistic Time-Dependent Schrödinger Equation

$$i \frac{\partial \psi(\vec{x},t)}{\partial t} = -\frac{1}{2m} \frac{\partial^2 \psi(\vec{x},t)}{\partial x^2} + \hat{V} \psi(\vec{x},t)$$

one-dimension

Probability Density and Probability Current

Max Born

The physical interpretation of the wavefunction $\psi(\vec{x},t)$

Probability of finding the particle in a volume $d^3\vec{x} = dx\,dy\,dz =$
$\psi^* \psi \, d^3\vec{x}$

The probability density $= \psi^* \psi$ $\rho(\vec{x},t) \equiv \psi^*(\vec{x},t)\,\psi(x,t)$

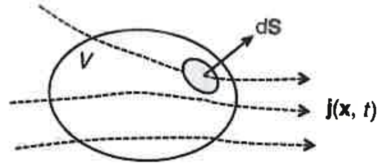
Conservation of Probability → in terms of the Continuity Equation

Let $\vec{j}(\vec{x},t)$ = flux density across an elemental surface $d\vec{S}$

The flux of probability across an elemental surface is: $\vec{j} \cdot d\vec{S}$

Rate of change of probability within V ↔ related to the net flux leaving the surface S.

$$\frac{\partial}{\partial t} \int_V \rho \, dV = -\int_S \vec{j} \cdot d\vec{S}$$



$$\frac{\partial}{\partial t} \int_V \rho \, dV = -\int_V \vec{\nabla} \cdot \vec{j}(x,t) \, dV$$

Continuity Equation $\boxed{\vec{\nabla} \cdot \hat{j} + \frac{\partial \rho}{\partial t} = 0}$

$\rho(\vec{x},t) = \psi^* \psi$

$\vec{j}(x,t) = \frac{1}{2mi} \left(\psi^* \nabla \psi - \psi \nabla \psi^* \right)$

$\vec{j} = |N|^2 \frac{\vec{P}}{m} \equiv \overset{\#/\text{volume}}{n} \vec{V}$

Chapter 2 Notes

Time Dependence and Conserved Quantities

Time dependence of an eigenstate of a Hamiltonian is:

$$\psi_i(\vec{x},t) = \underbrace{\phi_i(\vec{x})}_{} e^{-jEt}$$

↑solution of the time-indep. Schrödinger Eq.

$$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle = \int \psi^{\dagger}\hat{A}\psi\, d^3\vec{x}$$

$$\frac{d}{dt}\langle A\rangle = i\langle[\hat{H},\hat{A}]\rangle$$

A does not change with time if $[\hat{H},\hat{A}] = 0$.

↳ A is a conserved quantity

"Constant of the motion"

Commutation Relations

Must be <u>one</u> of two possibilities:

1.) $[\hat{A},\hat{B}] = 0$ There exists an eigenstate $|\varphi\rangle$ of both $\hat{A}$ and $\hat{B}$. $\hat{A}|\varphi\rangle = a|\varphi\rangle$ and $\hat{B}|\varphi\rangle = b|\varphi\rangle$

2.) $[\hat{A},\hat{B}] \neq 0$ It is not possible to define a simultaneous eigenstate of the operators $\hat{A}$ and $\hat{B}$

The limit to which the physical observables A and B can be known (simultaneously):

$$\boxed{(\Delta A)(\Delta B) \geq \frac{1}{2}\left|\langle i[\hat{A},\hat{B}]\rangle\right|}$$

where $(\Delta A)^2 = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$

Chapter 2 Notes

Position - Momentum Uncertainty Relation

$$\hat{x}\psi = x\psi \qquad \hat{p}_x\psi = -i\frac{\partial}{\partial x}\psi$$

$$[\hat{x},\hat{p}]\psi = i\psi \quad (\hbar = 1)$$

$$\Delta x\,\Delta p_x \geq \frac{1}{2}\left|\langle i[x,p]\rangle\right| \qquad \geq \frac{1}{2}\hbar$$

to make it dimensionally correct.

Angular Momentum in Quantum Mechanics

Ang. Momentum $\vec{L} = \vec{r}\times\vec{p} = L_x\hat{i} + L_y\hat{j} + L_z\hat{k}$

$$\hat{L}_x = (\hat{y}\hat{p}_z - \hat{z}\hat{p}_y) \qquad \hat{L}_y = (\hat{z}\hat{p}_x - \hat{x}\hat{p}_z) \qquad \hat{L}_z = (\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)$$

We have the following commutation relations:

$$[x,p_x] = [y,p_y] = [z,p_z] = i\hbar$$

$$[L_x,L_y] = iL_z\hbar \qquad [L_y,L_z] = iL_x\hbar \qquad [L_z,L_x] = iL_y\hbar$$

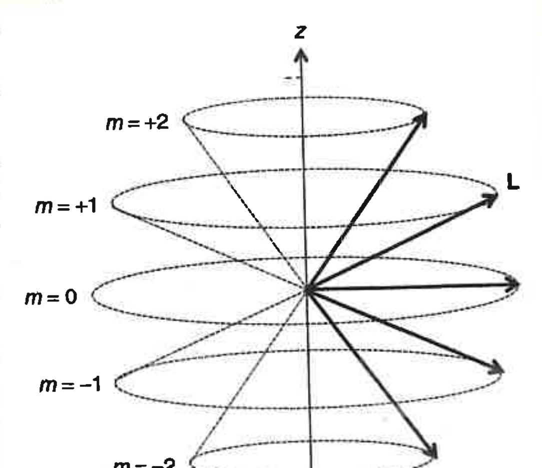
$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \quad \text{and} \quad [\hat{L}^2,\hat{L}_x] = [\hat{L}^2,\hat{L}_y] = [\hat{L}^2,\hat{L}_z] = 0$$

We have to pick either $\hat{L}_x$, $\hat{L}_y$, or $\hat{L}_z$ to simultaneously commute with $\hat{L}^2$. By convention, we have chosen the eigenfunctions that simultaneously commute with $\hat{L}^2$ and $\hat{L}_z$

$$[\hat{L}^2,\hat{L}_z] = 0 \qquad \hat{L}^2|\ell m\rangle = \ell(\ell+1)\hbar^2|\ell m\rangle$$

$$L_z|\ell m\rangle = m\hbar|\ell m\rangle$$

spherical harmonics $Y_\ell^m(\theta,\varphi)$



⇐ Spatial Quantization

Chapter 2 Notes

Fermi's Golden Rule

Particle Physics

Decay Rates and Scattering Cross Sections → in Q.M. means transitions between states

The equation to solve is ⇒ $$i\frac{d\psi}{dt} = \left[\hat{H}_0 + \hat{H}'(\vec{x},t)\right]\psi$$

$$\psi(\vec{x},t) = \sum_k c_k(t)\,\phi_k(\vec{x})\,e^{-iEt}$$

$$\frac{dc_f}{dt} = -i\langle f|\hat{H}'|i\rangle e^{i(E_f-E_i)t}$$

1^{st} order perturbation theory ⇒ $$\langle f|\hat{H}'|i\rangle = \int_V \phi_f^*(\vec{x})\,\hat{H}'\,\phi_i(\vec{x})\,d^3x$$

Transition Matrix Element

$$T_{fi} = \langle f|\hat{H}'|i\rangle$$

Probability for a transition to the state $\langle f|$ is:

$$P_{fi} = c_f(T)c_f^*(T) = |T_{fi}|^2\int_0^T dt'\int_0^T dt\, e^{i(E_f-E_i)t}\, e^{-i(E_f-E)t'}$$

$$d\Gamma_{fi} = \frac{P_{fi}}{T} \quad\Rightarrow\quad \underset{\substack{\uparrow\\ \text{Total Transition}\\ \text{Rate}}}{\Gamma_{fi}} = \frac{2\pi}{\hbar}|T_{fi}|^2 \underbrace{\left|\frac{dn}{dE}\right|_{E_i}}_{\text{Density of States}} = \rho(E_i)$$

$$\boxed{\Gamma_{fi} = 2\pi\,|T_{fi}|^2\rho(E_i)}$$ Fermi's Golden Rule